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Potentially $K_{r_1,r_2,\ldots,r_l,r,s}$-graphic sequences ☆

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Abstract

A variation of a classical Turán-type extremal problem is considered as follows: determine the smallest even integer $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geqslant \sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n)$ has a realization G containing $K_{r_1,r_2,\ldots,r_\ell,r,s}$ as a subgraph. In this paper, we determine $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n)$ for sufficiently large n, where $s \geqslant r \geqslant r_\ell \geqslant \cdots \geqslant r_1 \geqslant 0$ and $r \geqslant 3$.

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers with $d_i \leqslant n - 1$ for each i is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic non-increasing sequences in NS_n is denoted by GS_n. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For given a graph H, a sequence $\pi \in GS_n$ is said to be *potentially* (resp. *forcibly*) *H-graphic* if there exists a realization of π containing H as a subgraph (resp. each realization of π contains H as a subgraph).

The classical Turán-type extremal problem is to determine the smallest even integer $ex(H, n)$ such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geqslant ex(H, n)$ is forcibly H-graphic. In 1999, Gould et al. [3] considered the following variation of the classical Turán-type extremal problem: determine the smallest even integer $\sigma(H, n)$ such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geqslant \sigma(H, n)$ is potentially H-graphic. In [3], they also determined $\sigma(K_{2,2}, n)$ for $n \geqslant 4$. Recently, [7–10] determined $\sigma(K_{r,s}, n)$ for $s \geqslant r \geqslant 1$ and sufficiently large n. In [9], Yin et al. first introduced the following notations. Let

$$f(r, s, n) = (2r - 2)n - (r - 1)r + \frac{s(s + 4n - 4r + 2)}{8},$$

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$$g(r,s,n)=\left(2r+\frac{s-5}{2}\right)n-(r-1)r+\frac{(s-1)(s+1-4r)}{8}+s,$$

$$h(r,s,n)=(r+s-2)n-\frac{(r-1)(2s-r)}{2},$$

and let

$$A_1=\{(r,s,n)|s \text{ is even and } f(r,s,n) \text{ is also even}\},$$

$$A_2=\{(r,s,n)|s \text{ is even and } f(r,s,n) \text{ is odd}\},$$

$$B_1=\{(r,s,n)|s \text{ is odd and } g(r,s,n) \text{ is even}\},$$

$$B_2=\{(r,s,n)|s \text{ is odd and } g(r,s,n) \text{ is also odd}\},$$

$$C_1=\{(r,s,n)|h(r,s,n) \text{ is even}\},$$

$$C_2=\{(r,s,n)|h(r,s,n) \text{ is odd}\}.$$

Then, they proved the following:

Theorem 1.1 (*Yin et al.* [9]). (1) *If* $3\leqslant r\leqslant s\leqslant 2r$, *s is even and* $n\geqslant[(r+s+1)^2/4]+3s^2-2s-6$, *then*

$$\sigma(K_{r,s},n)=\begin{cases} f(r,s,n)+2 & \text{if } (r,s,n)\in A_1,\\ f(r,s,n)+1 & \text{if } (r,s,n)\in A_2.\end{cases}$$

(2) *If* $3\leqslant r\leqslant s\leqslant 2r$, *s is odd and* $n\geqslant[(r+s+1)^2/4]+3s^2+2s-8$, *then*

$$\sigma(K_{r,s},n)=\begin{cases} g(r,s,n)+2 & \text{if } (r,s,n)\in B_1,\\ g(r,s,n)+1 & \text{if } (r,s,n)\in B_2.\end{cases}$$

(3) *If* $r\geqslant 3$, $s\geqslant 2r+1$ *and* $n\geqslant[(r+s+1)^2/4]+2s^2+(2r-6)s+4r-8$, *then*

$$\sigma(K_{r,s},n)=\begin{cases} h(r,s,n)+2 & \text{if } (r,s,n)\in C_1,\\ h(r,s,n)+1 & \text{if } (r,s,n)\in C_2.\end{cases}$$

The purpose of this paper is to determine $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)$ for sufficiently large n, where $s\geqslant r\geqslant r_\ell\geqslant\cdots\geqslant r_1\geqslant 0$ and $r\geqslant 3$. In other words, we will prove the following:

Theorem 1.2. *Let* $s\geqslant r\geqslant r_\ell\geqslant\cdots\geqslant r_1\geqslant 0$, $r\geqslant 3$ *and n be sufficiently large. Denote* $m=r_1+r_2+\cdots+r_\ell$.
(1) *If* $3\leqslant r\leqslant s\leqslant 2r$ *and s is even, then*

$$\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)=\begin{cases} 2mn-m(m+1)+f(r,s,n-m)+2 & \text{if } (r,s,n-m)\in A_1,\\ 2mn-m(m+1)+f(r,s,n-m)+1 & \text{if } (r,s,n-m)\in A_2.\end{cases}$$

(2) *If* $3\leqslant r\leqslant s\leqslant 2r$ *and s is odd, then*

$$\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)=\begin{cases} 2mn-m(m+1)+g(r,s,n-m)+2 & \text{if } (r,s,n-m)\in B_1,\\ 2mn-m(m+1)+g(r,s,n-m)+1 & \text{if } (r,s,n-m)\in B_2.\end{cases}$$

(3) *If* $r\geqslant 3$ *and* $s\geqslant 2r+1$, *then*

$$\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)=\begin{cases} 2mn-m(m+1)+h(r,s,n-m)+2 & \text{if } (r,s,n-m)\in C_1,\\ 2mn-m(m+1)+h(r,s,n-m)+1 & \text{if } (r,s,n-m)\in C_2.\end{cases}$$

Remark 1. In the statement of Theorem 1.2, the hypothesis that “n be sufficiently large” does not mean that n has no a lower bound. In fact, by the proof of Theorem 1.2, we can choose the suitable positive integers c_1, c_2, c_3 and c_4, and create a lower bound of n as follows: $n\geqslant c_1(m+r+s+1)^4+c_2(m+r+s+1)^3+c_3(m+r+s+1)^2+c_4(m+r+s+1)$.

2. The lower bound for $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n)$

In order to prove the lower bound of $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n)$, we need the following Theorem 2.1 and Lemma 2.1.

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence. Denote $f'(\pi) = \max\{i \mid d_i \geqslant i\}$ and define an n-by-n matrix $A = (a_{ij})$ as follows: if $d_i \geqslant i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leqslant j \leqslant d_i + 1 \text{ and } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and if $d_i < i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leqslant j \leqslant d_i, \\ 0 & \text{otherwise.} \end{cases}$$

$f'(\pi)$ and A are called the *trace* and the *left-most off-diagonal matrix* of π, respectively. The column sum vector of A, denoted by $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$, is called the *corrected conjugate vector* of π. Clearly, the row sum vector of A is π and $\sigma(\overline{\pi}) = \sigma(\pi)$.

Theorem 2.1 (Berge [1]). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence with even $\sigma(\pi)$. Then π is graphic if and only if $d_1 + d_2 + \cdots + d_i \leqslant \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for each $i = 1, 2, \ldots, f'(\pi)$.*

For given $r_1, \ldots, r_\ell, r, s$ and n, where $s \geqslant r \geqslant r_\ell \geqslant \cdots \geqslant r_1 \geqslant 0$, $r \geqslant 3$, $m = r_1 + r_2 + \cdots + r_\ell$ and $n \geqslant m + r + s$, we define $\pi'(m, r, s, n)$ as follows. Note that the symbol x^y in a sequence stands for y consecutive terms, each equal to x.

If $3 \leqslant r \leqslant s \leqslant 2r$ and s is even, let

$$\pi'(m,r,s,n) = \begin{cases} \left(s-1, s-2, \ldots, \dfrac{s}{2}+1, \left(\dfrac{s}{2}\right)^{n-m-r-\frac{s}{2}+2}\right) & \text{if } (r,s,n-m) \in A_1, \\ \left(s-1, s-2, \ldots, \dfrac{s}{2}+1, \left(\dfrac{s}{2}\right)^{n-m-r-\frac{s}{2}+1}, \dfrac{s}{2}-1\right) & \text{if } (r,s,n-m) \in A_2. \end{cases}$$

If $3 \leqslant r \leqslant s \leqslant 2r$ and s is odd, let

$$\pi'(m,r,s,n) = \begin{cases} \left(s-1, \ldots, \dfrac{s}{2}+\dfrac{3}{2}, \left(\dfrac{s}{2}+\dfrac{1}{2}\right)^{\frac{s}{2}+\frac{3}{2}}, \left(\dfrac{s}{2}-\dfrac{1}{2}\right)^{n-m-r-s+1}\right) & \text{if } (r,s,n-m) \in B_1, \\ \left(s-1, \ldots, \dfrac{s}{2}+\dfrac{3}{2}, \left(\dfrac{s}{2}+\dfrac{1}{2}\right)^{\frac{s}{2}+\frac{3}{2}}, \left(\dfrac{s}{2}-\dfrac{1}{2}\right)^{n-m-r-s}, \dfrac{s}{2}-\dfrac{3}{2}\right) & \text{if } (r,s,n-m) \in B_2. \end{cases}$$

If $r \geqslant 3$ and $s \geqslant 2r+1$, let

$$\pi'(m,r,s,n) = \begin{cases} (s-1, s-2, \ldots, s-r+1, (s-r)^{n-m-2r+2}) & \text{if } (r,s,n-m) \in C_1, \\ (s-1, s-2, \ldots, s-r+1, (s-r)^{n-m-2r+1}, s-r-1) & \text{if } (r,s,n-m) \in C_2. \end{cases}$$

Lemma 2.1. (1) *$\pi'(m, r, s, n)$ is an $(n-m-r+1)$-term graphic sequence.*

(2) *For any positive integers r' and s', $1 \leqslant r' \leqslant \min\left\{r, \left[\frac{s}{2}+\frac{1}{2}\right]\right\}$ and $r' + s' = s+1$, $\pi'(m, r, s, n)$ is not potentially $K_{r',s'}$-graphic.*

Proof. (1) By the definitions of $\pi'(m, r, s, n)$ and $f'(\pi)$, it is easy to see that $\sigma(\pi'(m, r, s, n))$ is even and

$$f'(\pi'(m,r,s,n)) = \begin{cases} \dfrac{s}{2} & \text{if } 3 \leqslant r \leqslant s \leqslant 2r \text{ and } s \text{ is even,} \\ \dfrac{s}{2}+\dfrac{1}{2} & \text{if } 3 \leqslant r \leqslant s \leqslant 2r \text{ and } s \text{ is odd,} \\ s-r & \text{if } r \geqslant 3 \text{ and } s \geqslant 2r+1. \end{cases}$$

Also, it is easy to see from the left-most off-diagonal matrix A of $\pi'(m,r,s,n)$ that $\overline{\pi'(m,r,s,n)}=(\overline{d_1},\overline{d_2},\ldots,\overline{d_{n-m-r+1}})$ satisfies that

$$\overline{d_1}=\cdots=\overline{d_{f'(\pi'(m,r,s,n))-2}}=n-m-r\geqslant s \quad\text{and}\quad \overline{d_{f'(\pi'(m,r,s,n))-1}},\quad \overline{d_{f'(\pi'(m,r,s,n))}}\geqslant s-1.$$

Clearly, $d_1+d_2+\cdots+d_i\leqslant\overline{d_1}+\overline{d_2}+\cdots+\overline{d_i}$ for $i=1,2,\ldots,f'(\pi'(m,r,s,n))$. By Theorem 2.1, $\pi'(m,r,s,n)$ is graphic.

(2) Assume that $\pi'(m,r,s,n)$ has a realization G' containing $K_{r',s'}$ as a subgraph, where $1\leqslant r'\leqslant\min\left\{r,\left[\frac{s}{2}+\frac{1}{2}\right]\right\}$ and $r'+s'=s+1$. If $r'<\frac{s}{2}+\frac{1}{2}$, then there are at least r' terms in the degree sequence $\pi(G')$ of G' which are greater than or equal to $s+1-r'$, which is impossible. If $r'=\frac{s}{2}+\frac{1}{2}$, then there are at least $s+1$ terms in the degree sequence $\pi(G')$ of G' which are greater than or equal to $\frac{s}{2}+\frac{1}{2}$, which is also impossible. Therefore, $\pi'(m,r,s,n)$ is not potentially $K_{r',s'}$-graphic. □

Remark 2. Let H be a realization of $\pi'(m,r,s,n)$. By Lemma 2.1, H contains no $K_{r',s'}$ as a subgraph, where $1\leqslant r'\leqslant\min\left\{r,\left[\frac{s}{2}+\frac{1}{2}\right]\right\}$ and $r'+s'=s+1$. Consider $K_{r-1}+H$, the join of K_{r-1} and H, is the graph with $V(K_{r-1}+H)=V(K_{r-1})\cup V(H)$ and $E(K_{r-1}+H)=E(K_{r-1})\cup E(H)\cup\{uv|u\in V(K_{r-1}),v\in V(H)\}$. It is easy to see that $K_{r-1}+H$ contains no $K_{r,s}$ and yields the functions $f(r,s,n-m)$, $g(r,s,n-m)$ and $h(r,s,n-m)$. In other words, $f(r,s,n-m)$, $g(r,s,n-m)$ and $h(r,s,n-m)$ are established by $\sigma(\pi(K_{r-1}+H))$, where $\pi(K_{r-1}+H)$ is the degree sequence of $K_{r-1}+H$. Specifically, if $m=0$, then $K_{r-1}+H$ is the graph given to establish the lower bound of $\sigma(K_{r,s},n)$ in [9], and hence yields the functions $f(r,s,n)$, $g(r,s,n)$ and $h(r,s,n)$. In order to establish the lower bound of $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)$, it is natural to consider the graph $K_m+(K_{r-1}+H)$, i.e., $K_{m+r-1}+H$. Now, we prove the following Theorem 2.2.

Theorem 2.2. *Let $s\geqslant r\geqslant r_\ell\geqslant\cdots\geqslant r_1\geqslant 0$, $r\geqslant 3$ and $n\geqslant m+r+s$, where $m=r_1+r_2+\cdots+r_\ell$.*
(1) *If $3\leqslant r\leqslant s\leqslant 2r$ and s is even, then*

$$\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)\geqslant\begin{cases}2mn-m(m+1)+f(r,s,n-m)+2 & \text{if } (r,s,n-m)\in A_1,\\ 2mn-m(m+1)+f(r,s,n-m)+1 & \text{if } (r,s,n-m)\in A_2.\end{cases}$$

(2) *If $3\leqslant r\leqslant s\leqslant 2r$ and s is odd, then*

$$\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)\geqslant\begin{cases}2mn-m(m+1)+g(r,s,n-m)+2 & \text{if } (r,s,n-m)\in B_1,\\ 2mn-m(m+1)+g(r,s,n-m)+1 & \text{if } (r,s,n-m)\in B_2.\end{cases}$$

(3) *If $r\geqslant 3$ and $s\geqslant 2r+1$, then*

$$\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)\geqslant\begin{cases}2mn-m(m+1)+h(r,s,n-m)+2 & \text{if } (r,s,n-m)\in C_1,\\ 2mn-m(m+1)+h(r,s,n-m)+1 & \text{if } (r,s,n-m)\in C_2.\end{cases}$$

Proof. We will construct a graph G which contains no $K_{r_1,r_2,\ldots,r_\ell,r,s}$ and whose degree sequence is not potentially $K_{r_1,r_2,\ldots,r_\ell,r,s}$-graphic. Let H be a realization of $\pi'(m,r,s,n)$ as guaranteed by Lemma 2.1 part (1). By Lemma 2.1 part (2), the degree sequence $\pi(H)$ of H is not potentially $K_{r',s'}$-graphic, where $1\leqslant r'\leqslant\min\left\{r,\left[s/2+\frac{1}{2}\right]\right\}$ and $r'+s'=s+1$. Take $G=K_{m+r-1}+H$. Then the degree sequence $\pi(G)$ of G and the degree sum $\sigma(\pi(G))$ of G, denoted by $\pi(m,r,s,n)$ and $\sigma(\pi(m,r,s,n))$, respectively, are determined by the parameters m, r, s and n, as follows.

(a) If $3\leqslant r\leqslant s\leqslant 2r$, s is even and $(r,s,n-m)\in A_1$, then

$$\pi(G)=\pi(m,r,s,n)=\left((n-1)^{m+r-1},m+r+s-2,m+r+s-3,\ldots,m+r+\frac{s}{2},\right.$$
$$\left.\left(m+r+\frac{s}{2}-1\right)^{n-m-r-s/2+2}\right),$$

$$\sigma(\pi(G))=\sigma(\pi(m,r,s,n))=2mn-m(m+1)+f(r,s,n-m).$$

(b) If $3\leqslant r\leqslant s\leqslant 2r$, s is even and $(r,s,n-m)\in A_2$, then

$$\pi(G)=\pi(m,r,s,n)=\Big((n-1)^{m+r-1},m+r+s-2,m+r+s-3,\ldots,m+r+\frac{s}{2},$$
$$\Big(m+r+\frac{s}{2}-1\Big)^{n-m-r-s/2+1},m+r+\frac{s}{2}-2\Big),$$
$$\sigma(\pi(G))=\sigma(\pi(m,r,s,n))=2mn-m(m+1)+f(r,s,n-m)-1.$$

(c) If $3\leqslant r\leqslant s\leqslant 2r$, s is odd and $(r,s,n-m)\in B_1$, then

$$\pi(G)=\pi(m,r,s,n)=\Big((n-1)^{m+r-1},m+r+s-2,m+r+s-3,\ldots,m+r+\frac{s}{2}+\frac{1}{2},$$
$$\Big(m+r+\frac{s}{2}-\frac{1}{2}\Big)^{s/2+3/2},\Big(m+r+\frac{s}{2}-\frac{3}{2}\Big)^{n-m-r-s+1}\Big),$$
$$\sigma(\pi(G))=\sigma(\pi(m,r,s,n))=2mn-m(m+1)+g(r,s,n-m).$$

(d) If $3\leqslant r\leqslant s\leqslant 2r$, s is odd and $(r,s,n-m)\in B_2$, then

$$\pi(G)=\pi(m,r,s,n)=\Big((n-1)^{m+r-1},m+r+s-2,m+r+s-3,\ldots,m+r+\frac{s}{2}+\frac{1}{2},$$
$$\Big(m+r+\frac{s}{2}-\frac{1}{2}\Big)^{s/2+3/2},\Big(m+r+\frac{s}{2}-\frac{3}{2}\Big)^{n-m-r-s},m+r+\frac{s}{2}-\frac{5}{2}\Big),$$
$$\sigma(\pi(G))=\sigma(\pi(m,r,s,n))=2mn-m(m+1)+g(r,s,n-m)-1.$$

(e) If $r\geqslant 3$, $s\geqslant 2r+1$ and $(r,s,n-m)\in C_1$, then

$$\pi(G)=\pi(m,r,s,n)=((n-1)^{m+r-1},m+r+s-2,m+r+s-3,\ldots,m+s,$$
$$(m+s-1)^{n-m-2r+2}),$$
$$\sigma(\pi(G))=\sigma(\pi(m,r,s,n))=2mn-m(m+1)+h(r,s,n-m).$$

(f) If $r\geqslant 3$, $s\geqslant 2r+1$ and $(r,s,n-m)\in C_2$, then

$$\pi(G)=\pi(m,r,s,n)=((n-1)^{m+r-1},m+r+s-2,m+r+s-3,\ldots,m+s,$$
$$(m+s-1)^{n-m-2r+1},m+s-2),$$
$$\sigma(\pi(G))=\sigma(\pi(m,r,s,n))=2mn-m(m+1)+h(r,s,n-m)-1.$$

Let G' be a realization of $\pi(m,r,s,n)$. It is easy to see from the degree sequence of G' that G' is a copy of K_{m+r-1} joined with another graph H' whose degree sequence is exactly $\pi(H)$. We now assume G' contains $K_{r_1,r_2,\ldots,r_\ell,r,s}$ as a subgraph. We also assume $K_{r_1,r_2,\ldots,r_\ell,r,s}$ has partition sets $D_1,D_2,\ldots,D_{\ell+2}$ in G'. There exist at least $s+1$ vertices in $K_{r_1,r_2,\ldots,r_\ell,r,s}$ that belong to H'. These vertices must lie in at least two partite sets, and hence form a complete bipartite graph $K_{r',s'}$, where $1\leqslant r'\leqslant \min\left\{r,\left[\frac{s}{2}+\frac{1}{2}\right]\right\}$ and $r'+s'=s+1$. In other words, $\pi(H')$ is potentially $K_{r',s'}$-graphic, where $1\leqslant r'\leqslant \min\left\{r,\left[\frac{s}{2}+\frac{1}{2}\right]\right\}$and $r'+s'=s+1$, a contradiction. Hence, $\pi(m,r,s,n)$ is not potentially $K_{r_1,r_2,\ldots,r_\ell,r,s}$-graphic. Thus, we have $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s},n)\geqslant\sigma(\pi(m,r,s,n))+2$, i.e., Theorem 2.2 holds. $\square$

3. Proof of Theorem 1.2

In this section, we always assume that $s \geqslant r \geqslant r_\ell \geqslant \cdots \geqslant r_1 \geqslant 0$, $r \geqslant 3$ and $m = r_1 + r_2 + \cdots + r_\ell$. Moreover, "$n$ be sufficiently large" just means that $n \geqslant c_1(m+r+s+1)^4 + c_2(m+r+s+1)^3 + c_3(m+r+s+1)^2 + c_4(m+r+s+1)$, where c_1, c_2, c_3 and c_4 are the suitable positive integers. In order to prove Theorem 1.2, we need the following known theorems.

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence and $1 \leqslant k \leqslant n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) & \text{if } d_k \geqslant k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) & \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \geqslant d'_2 \geqslant \cdots \geqslant d'_{n-1}$ is the rearrangement of the $n-1$ terms of π''_k. Then π'_k is called the *residual sequence* obtained from π by laying off d_k. It is easy to see that if π'_k is graphic then so is π, since a realization G of π can be obtained from a realization G' of π'_k by adding a new vertex of degree d_k and joining it to the vertices whose degrees are reduced by one in going from π to π'_k. In fact more is true:

Theorem 3.1 (*Kleitman and Wang [5]*). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence and $1 \leqslant k \leqslant n$. Then π is graphic if and only if π'_k is graphic.*

Theorem 3.2 (*Erdős and Gallai [2]*). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence with even $\sigma(\pi)$. Then π is graphic if and only if for any t, $1 \leqslant t \leqslant n-1$,*

$$\sum_{i=1}^{t} d_i \leqslant t(t-1) + \sum_{j=t+1}^{n} \min\{t, d_j\}.$$

Theorem 3.3 (*Yin and Li [7,8]*). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $x = \max\{d_1, d_2, \ldots, d_n\}$ and $\sigma(\pi)$ be even. Let $\pi^* = (d^*_1, d^*_2, \ldots, d^*_n)$ be the rearrangement sequence of π, where $x = d^*_1 \geqslant d^*_2 \geqslant \cdots \geqslant d^*_n$ is the rearrangement of $d_1, d_2, \ldots, d_n$. If there exists an integer n_1, $n_1 \leqslant n$ such that $d^*_{n_1} \geqslant y \geqslant 1$ and $n_1 \geqslant \frac{1}{y}\left[\frac{(x+y+1)^2}{4}\right]$, then π is graphic.*

If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is *potentially A_{r+1}-graphic*. In [6], Rao gave a necessary and sufficient condition for a sequence $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ to be potentially A_{r+1}-graphic. A simple sufficient condition was also proved. The following are his results.

Theorem 3.4 (*Rao [6]*). *Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. Then π is potentially A_{r+1}-graphic if and only if the following conditions hold*:

(i) $d_{r+1} \geqslant r$,
(ii) *$\sigma(\pi)$ is even,*
(iii) *for any s and t, $0 \leqslant s \leqslant r+1$ and $0 \leqslant t \leqslant n-r-1$,*

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \leqslant (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} \min\{s+t, d_i - r + s\} + \sum_{i=r+t+2}^{n} \min\{s+t, d_i\}.$$

Theorem 3.5 (*Rao [6]*). *Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. If $d_{r+1} \geqslant 2r - 1$, then π is potentially A_{r+1}-graphic.*

The original proofs of Theorems 3.4 and 3.5 remain unpublished, but Kézdy and Lehel [4] have given different proofs using network flows.

Theorem 3.6 (*Gould et al. [3]*). *If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

In order to prove Theorem 1.2, we also need the following lemmas. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, we denote $p(\pi) = \max\{i \mid d_i \geqslant 1\}$.

Lemma 3.1. *Let n be sufficiently large and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n)) + 2$. Then*

(1) $d_m \geqslant m + r + s - 1$.
(2) $d_{m+r+s} \geqslant m + r$.
(3) $p(\pi) \geqslant \sqrt{\sigma(\pi)}$.

Proof. (1) If $d_m \leqslant m+r+s-2$, then $\sigma(\pi) = \sum_{i=1}^{n} d_i \leqslant (m-1)(n-1) + (n-m+1)(m+r+s-2) < \sigma(\pi(m, r, s, n)) + 2$, a contradiction. Hence, $d_m \geqslant m + r + s - 1$.

(2) If $d_{m+r+s} \leqslant m+r-1$, then by Theorem 3.2, $\sigma(\pi) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{m+r+s-1} d_i + \sum_{i=m+r+s}^{n} d_i \leqslant ((m+r+s-1)(m+r+s-2) + \sum_{i=m+r+s}^{n} \min\{m+r+s-1, d_i\}) + \sum_{i=m+r+s}^{n} d_i = (m+r+s-1)(m+r+s-2) + 2\sum_{i=m+r+s}^{n} d_i \leqslant 2(n-m-r-s+1)(m+r-1) + (m+r+s-1)(m+r+s-2) = (2m+2r-2)n - (m+r+s-1)(m+r-s) < \sigma(\pi(m, r, s, n)) + 2$, a contradiction. Hence, $d_{m+r+s} \geqslant m + r$.

(3) Since $(p(\pi))^2 \geqslant p(\pi)(p(\pi) - 1) \geqslant p(\pi) d_1 \geqslant \sum_{i=1}^{n} d_i = \sigma(\pi)$, we have $p(\pi) \geqslant \sqrt{\sigma(\pi)}$. □

Lemma 3.2. *Let n be sufficiently large and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n)) + 2$ and $d_{m+r} \leqslant m + r + s - 2$. Then there exists an integer t, $1 \leqslant t \leqslant \min\{r - 1, \lceil \frac{s}{2} \rceil - 1\}$ such that $d_{m+r+t} \geqslant m + r + s - 1 - t$ and $d_{m+r+s} \geqslant m + r + t$.*

Proof. *Case* 1: $3 \leqslant r \leqslant s \leqslant 2r$ and s is even. If $d_{m+r+t} \leqslant m + r + s - 2 - t$ for any $t \in \{1, 2, \ldots, \frac{s}{2} - 1\}$, then $\sigma(\pi) \leqslant (m + r - 1)(n - 1) + (m + r + s - 2) + (m + r + s - 3) + \cdots + (m + r + \frac{s}{2}) + (m + r + \frac{s}{2} - 1)(n - m - r - \frac{s}{2} + 2) = 2mn - m(m + 1) + f(r, s, n - m)$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, \frac{s}{2} - 1\}$ such that $d_{m+r+t} \geqslant m + r + s - 1 - t$. If $d_{m+r+s} \leqslant m + r + \frac{s}{2} - 2$, then $\sigma(\pi) \leqslant (m + r - 1)(n - 1) + s(m + r + s - 2) + (m + r + \frac{s}{2} - 2)(n - m - r - s + 1) < 2mn - m(m + 1) + f(r, s, n - m) + 1$, a contradiction. Hence $d_{m+r+s} \geqslant m + r + \frac{s}{2} - 1 \geqslant m + r + t$.

Case 2: $3 \leqslant r \leqslant s \leqslant 2r$ and s is odd. If $d_{m+r+t} \leqslant m + r + s - 2 - t$ for any $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{1}{2}\}$, then $\sigma(\pi) \leqslant (m + r - 1)(n - 1) + (m + r + s - 2) + (m + r + s - 3) + \cdots + (m + r + \frac{s}{2} - \frac{1}{2}) + (m + r + \frac{s}{2} - \frac{3}{2})(n - m - r - \frac{s}{2} + \frac{3}{2}) < 2mn - m(m + 1) + g(r, s, n - m) + 1$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{1}{2}\}$ such that $d_{m+r+t} \geqslant m + r + s - 1 - t$. There are two subcases.

Subcase 1: There exists an integer $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{3}{2}\}$ such that $d_{m+r+t} \geqslant m+r+s-1-t$. If $d_{m+r+s} \leqslant m+r+\frac{s}{2}-\frac{5}{2}$, then $\sigma(\pi) \leqslant (m+r-1)(n-1) + s(m+r+s-2) + (m+r+\frac{s}{2}-\frac{5}{2})(n-m-r-s+1) < 2mn - m(m+1) + g(r, s, n-m) + 1$, a contradiction. Hence $d_{m+r+s} \geqslant m + r + s/2 - \frac{3}{2} \geqslant m + r + t$.

Subcase 2: $d_{m+r+t} \leqslant m + r + s - 2 - t$ for any $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{3}{2}\}$ and $d_{m+r+t} \geqslant m + r + s - 1 - t$ for $t = \frac{s}{2} - \frac{1}{2}$. If $d_{m+r+s} \leqslant m + r + \frac{s}{2} - \frac{3}{2}$, then $\sigma(\pi) \leqslant (m + r - 1)(n - 1) + (m + r + s - 2) + (m + r + s - 3) + \cdots + (m + r + \frac{s}{2} + \frac{1}{2}) + (m + r + \frac{s}{2} - \frac{1}{2})(\frac{s}{2} + \frac{3}{2}) + (m + r + \frac{s}{2} - \frac{3}{2})(n - m - r - s + 1) = 2mn - m(m + 1) + g(r, s, n - m)$, which is impossible. Hence $d_{m+r+s} \geqslant m + r + \frac{s}{2} - \frac{1}{2} = m + r + t$.

Case 3: $r \geqslant 3$ and $s \geqslant 2r + 1$. If $d_{m+r+t} \leqslant m + r + s - 2 - t$ for any $t \in \{1, 2, \ldots, r - 1\}$, then $\sigma(\pi) \leqslant (m + r - 1)(n - 1) + (m + r + s - 2) + (m + r + s - 3) + \cdots + (m + s) + (m + s - 1)(n - m - 2r + 2) = 2mn - m(m + 1) + h(r, s, n - m)$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, r - 1\}$ such that $d_{m+r+t} \geqslant m + r + s - 1 - t$. If $d_{m+r+s} \leqslant m + 2r - 2$, then by $s \geqslant 2r + 1$, $\sigma(\pi) \leqslant (m + r - 1)(n - 1) + s(m + r + s - 2) + (m + 2r - 2)(n - m - r - s + 1) < 2mn - m(m + 1) + h(r, s, n - m) + 1$, a contradiction. Hence $d_{m+r+s} \geqslant m + 2r - 1 \geqslant m + r + t$. □

Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$. If π has a realization H with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ such that (1) $d_H(v_i) = d_i$ for $1 \leqslant i \leqslant n$, (2) H contains $K_{r,s}$ as its subgraph, where $\{v_1, \ldots, v_r\}, \{u_1, \ldots, u_s\} (\subseteq \{v_{r+1}, \ldots, v_n\})$ is the bipartite partition of the vertex set of $K_{r,s}$, (3) the induced subgraph of $\{v_1, \ldots, v_r\}$ in H is a clique, then π is said to be *potentially* $B_{r,s}$-graphic. Furthermore, if $\{u_1, \ldots, u_s\} = \{v_{r+1}, \ldots, v_{r+s}\}$, then π is said to be *potentially* $A_{r,s}$-graphic. It is easy to see that if π is potentially $A_{r,s}$-graphic, then π is potentially $B_{r,s}$-graphic. The following Lemma 3.3 shows that the converse is also true.

Lemma 3.3. *If $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ is potentially $B_{r,s}$-graphic,then π is potentially $A_{r,s}$-graphic.*

Proof. Since π is potentially $B_{r,s}$-graphic, we can choose a realization H with vertex set $V(H)=\{v_1,v_2,\ldots,v_n\}$ such that (1) $d_H(v_i)=d_i$ for $1\leqslant i\leqslant n$, (2) H contains $K_{r,s}$ as its subgraph, where $X=\{v_1,\ldots,v_r\}$, $Y=\{u_1,\ldots,u_s\}(\subseteq\{v_{r+1},\ldots,v_n\})$ is the bipartite partition of the vertex set of $K_{r,s}$, (3) the induced subgraph of $\{v_1,\ldots,v_r\}$ in H is a clique, (4) $|Z\cap Y|$ is maximum, where $Z=\{v_{r+1},\ldots,v_{r+s}\}$. Clearly, if $|Z\cap Y|=s$, i.e., $Z=Y$, then π is potentially $A_{r,s}$-graphic. If $|Z\cap Y|<s$, then $Y-Z$ and $Z-Y$ are nonempty. Hence there are $v\in Z-Y$ and $u\in Y-Z$ such that $d_H(v)\geqslant d_H(u)$. By the choice of H and Y, we have $X\nsubseteq N(v)$, where $N(v)$ is the neighbor set of v in H. Now assume that $X\cap N(v)=\{v_1,\ldots,v_t\}$, where $0\leqslant t\leqslant r-1$. Since $d_H(v)\geqslant d_H(u)$, $|N(v)-(X\cup\{u\})|\geqslant|N(u)-(X\cup\{v\})|+r-t$. Hence, there exist $w_1,\ldots,w_{r-t}\in N(v)-(X\cup\{u\})$ such that $w_iu\notin E(H)$ for $1\leqslant i\leqslant r-t$. Let

$$H'=H-\{v_iu|i=t+1,\ldots,r\}+\{w_iu|i=1,\ldots,r-t\}$$
$$-\{w_iv|i=1,\ldots,r-t\}+\{v_iv|i=t+1,\ldots,r\}.$$

Then H' is a realization of π, and contains $K_{r,s}$ as its subgraph, where X, $Y-\{u\}+\{v\}$ is the bipartite partition of the vertex set of $K_{r,s}$ and the induced subgraph of X in H' is still a clique. Clearly, $|(Y-\{u\}+\{v\})\cap Z|>|Z\cap Y|$, which is impossible since $|Z\cap Y|$ is maximum. $\square$

For $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$, if π has a realization H with vertex set $V(H)=\{v_1,v_2,\ldots,v_n\}$ such that (1) $d_H(v_i)=d_i$ for $1\leqslant i\leqslant n$, (2) H contains $K_{t,k,h}$ as its subgraph, where $\{v_1,\ldots,v_t\}$, $\{u_1,\ldots,u_k\}(\subseteq\{v_{t+1},\ldots,v_{t+k+h}\})$, $\{w_1,\ldots,w_h\}(\subseteq\{v_{t+1},\ldots,v_{t+k+h}\})$ is the 3-partite partition of the vertex set of $K_{t,k,h}$, (3) the induced subgraph of $\{v_1,\ldots,v_t\}$ in H is a clique, then π is said to be *potentially $B_{t,k,h}$-graphic*. Furthermore, if $\{u_1,\ldots,u_k\}=\{v_{t+1},\ldots,v_{t+k}\}$ and $\{w_1,\ldots,w_h\}=\{v_{t+k+1},\ldots,v_{t+k+h}\}$, then π is said to be *potentially $A_{t,k,h}$-graphic*.

Lemma 3.4. *Let $m\leqslant t\leqslant m+r+s$ and $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ with $d_t\geqslant m+r+s-1$, $d_{m+r+s}\geqslant t$ and $n-2\geqslant d_1\geqslant\cdots\geqslant d_{m+r+s-1}\geqslant d_{m+r+s}=\cdots=d_{d_1+2}\geqslant d_{d_1+3}\geqslant\cdots\geqslant d_n$. We define sequences $\pi_0,\ldots,\pi_t$ as follows. Let $\pi_0=\pi$. Let*

$$\pi_1=(d_2-1,\ldots,d_{m+r+s}-1,d^{(1)}_{m+r+s+1},\ldots,d^{(1)}_n),$$

where $d^{(1)}_{m+r+s+1}\geqslant\cdots\geqslant d^{(1)}_n$ is the rearrangement of $d_{m+r+s+1}-1,\ldots,d_{d_1+1}-1,d_{d_1+2},\ldots,d_n$. For $2\leqslant i\leqslant t$, given $\pi_{i-1}=(d_i-i+1,\ldots,d_{m+r+s}-i+1,d^{(i-1)}_{m+r+s+1},\ldots,d^{(i-1)}_n)$, let

$$\pi_i=(d_{i+1}-i,\ldots,d_{m+r+s}-i,d^{(i)}_{m+r+s+1},\ldots,d^{(i)}_n),$$

where $d^{(i)}_{m+r+s+1}\geqslant\cdots\geqslant d^{(i)}_n$ is the rearrangement of $d^{(i-1)}_{m+r+s+1}-1,\ldots,d^{(i-1)}_{d_i+1}-1,d^{(i-1)}_{d_i+2},\ldots,d^{(i-1)}_n$. Then, we have

(1) *$\sigma(\pi_i)$ is even for $1\leqslant i\leqslant t$,*
(2) *if $d^{(t)}_{m+r+s+1}\geqslant 2$, then $p(\pi_t)=p(\pi)$,*
(3) *if π_t is graphic, then π is potentially $A_{t,m+r+s-t}$-graphic.*

Proof. (1) is obvious.

(2) For each $\pi_i=(d_{i+1}-i,\ldots,d_{m+r+s}-i,d^{(i)}_{m+r+s+1},\ldots,d^{(i)}_n)$, denote $k_i=\max\{j\mid d^{(i)}_{m+r+s+1}-d^{(i)}_{m+r+s+j}\leqslant 1\}$. Clearly, $m+r+s+k_0\geqslant d_1+2$, and $d^{(i-1)}_{m+r+s+1}-d^{(i-1)}_{m+r+s+k_{i-1}}\leqslant 1$ implies that $d^{(i)}_{m+r+s+1}-d^{(i)}_{m+r+s+k_{i-1}}\leqslant 1$ for $1\leqslant i\leqslant t$. Hence $k_t\geqslant k_{t-1}\geqslant\cdots\geqslant k_0\geqslant d_1+2-(m+r+s)$. Since $\min\{d^{(i-1)}_{m+r+s+1}-1,\ldots,d^{(i-1)}_{d_i+1}-1,d^{(i-1)}_{d_i+2},\ldots,d^{(i-1)}_{m+r+s+k_{i-1}}\}\geqslant d^{(i-1)}_{m+r+s+1}-2\geqslant d^{(i-1)}_{m+r+s+k_{i-1}+1}\geqslant\cdots\geqslant d^{(i-1)}_n$, we have $d^{(i)}_{m+r+s+k_{i-1}+j}=d^{(i-1)}_{m+r+s+k_{i-1}+j}$ for $j\geqslant 1$. Thus $d^{(i)}_{m+r+s+k}=d^{(i-1)}_{m+r+s+k}$ for $k>k_i$. Consequently, $d^{(t)}_{m+r+s+k}=d_{m+r+s+k}$ for $k>k_t$. If $m+r+s+k_t<p(\pi)$, i.e., $k_t<p(\pi)-(m+r+s)$, then $p(\pi_t)=p(\pi)$. If $m+r+s+k_t=p(\pi)$, i.e., $k_t=p(\pi)-(m+r+s)$, then by $d^{(t)}_{m+r+s+1}\geqslant 2$, we have $d^{(t)}_{m+r+s+k_t}\geqslant 1$. In other words, $d^{(t)}_{p(\pi)}\geqslant 1$. Thus, $p(\pi_t)$ is also $p(\pi)$.

(3) follows immediately from the definition of π_t. $\square$

Lemma 3.5. *Let n be sufficiently large and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n)) + 2$ and $n - 2 \geqslant d_1 \geqslant \cdots \geqslant d_{m+r+s-1} \geqslant d_{m+r+s} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n$. If $d_{m+r+s} \geqslant m + r + s - 1$, then π is potentially $A_{m,r,s}$-graphic.*

Proof. If $d_{m+r+s} \geqslant 2(m+r+s)-3$, then by Theorem 3.5, π is potentially A_{m+r+s}-graphic, and so π is potentially $A_{m,r,s}$-graphic. Assume that $m+r+s-1 \leqslant d_{m+r+s} \leqslant 2(m+r+s)-4$. Take $t=m+r+s$. Then $\pi_t=(d^{(t)}_{m+r+s+1}, \dots, d^{(t)}_n)$. Clearly, $d^{(t)}_{m+r+s+1} \leqslant d_{m+r+s+1} \leqslant 2(m+r+s)-4$. If $d^{(t)}_{m+r+s+1} \geqslant 2$, then by Lemmas 3.4(2) and 3.1(3), $p(\pi_t) = p(\pi) \geqslant \sqrt{\sigma(\pi)}$ is sufficiently large. So $p(\pi_t) - (m + r + s) \geqslant \frac{1}{y}\left[\frac{(x+y+1)^2}{4}\right]$,where $x = 2(m + r + s) - 4$ and $y = 1$. By Theorem 3.3, π_t is graphic. If $d^{(t)}_{m+r+s+1} = 1$, then by Lemma 3.4(1), π_t is clearly graphic. Thus, π is potentially $A_{m+r+s,0}$-graphic by Lemma 3.4(3). π is also potentially $A_{m,r,s}$-graphic. □

Lemma 3.6. *Let $\pi=(d_1, d_2, \dots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n))+2$, $n-2 \geqslant d_1 \geqslant \cdots \geqslant d_{m+r+s-1} \geqslant d_{m+r+s} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n$ and $d_{m+r+s} \leqslant m+r+s-2$. If $d_{m+r} \geqslant m+r+s+\max\{m+r, s\}$, then π is potentially $A_{m,r,s}$-graphic.*

Proof. Let $\rho_0 = \pi$. For $i = 1, 2, \dots, s$ in turn, let ρ_i be the residual sequence obtained from ρ_{i-1} by laying off the $(m + r + 1)$th term. By $d_{m+r} \geqslant m + r + s + \max\{m + r, s\}$ and $d_{m+r+s} \leqslant m + r + s - 2$, it is easy to see that $d_1 - i$, $d_2 - i, \dots, d_{m+r} - i$ are the $m + r$ largest terms in ρ_i for $i = 0, 1, \dots, s$ in turn. Theorem 3.1 implies that ρ_i is graphic for $i = 0, 1, \dots, s$. Since the $(m + r)$th largest term in ρ_s is $d_{m+r} - s$ ($\geqslant 2(m + r)$), by Theorem 3.5,ρ_s is potentially A_{m+r}-graphic. Thus, it is easy to get from Theorem 3.1 that ρ_i is potentially $B_{m+r,s-i}$-graphic for $i = s-1, s-2, \dots, 0$ in turn. Specifically, when $i = 0$, π is potentially $B_{m+r,s}$-graphic. By Lemma 3.3, π is potentially $A_{m+r,s}$-graphic. π is also potentially $A_{m,r,s}$-graphic. □

Lemma 3.7. *Let n be sufficiently large and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n)) + 2$, $n - 2 \geqslant d_1 \geqslant \cdots \geqslant d_{m+r+s-1} \geqslant d_{m+r+s} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n$ and $d_{m+r+s} \leqslant m + r + s - 2$. If $m + r + s - 1 \leqslant d_{m+r} \leqslant m + r + s - 1 + \max\{m + r, s\}$, then π is potentially $A_{m,r,s}$-graphic.*

Proof. Take $t = m + r$. By Lemmas 3.1(2) and 3.4, $d_{m+r+s} \geqslant t$ and $\pi_t = (d_{m+r+1} - (m + r), \dots, d_{m+r+s} - (m + r), d^{(t)}_{m+r+s+1}, \dots, d^{(t)}_n)$. Clearly, $d_{m+r+1} - (m + r) \leqslant s - 1 + \max\{m + r, s\}$ and $1 \leqslant d^{(t)}_{m+r+s+1} \leqslant m + r + s - 2$. If $d^{(t)}_{m+r+s+1} \geqslant 2$, then by Lemmas 3.4(2) and 3.1(3), $p(\pi_t) = p(\pi) \geqslant \sqrt{\sigma(\pi)}$ is sufficiently large. If $d^{(t)}_{m+r+s+1} = 1$, then $p(\pi_t)$ is also sufficiently large since $d_{m+r} - (m + r + s - 1) \leqslant \max\{m + r, s\}$. Now it is easy to follow from Theorem 3.3 that π_t is graphic. Thus, π is potentially $A_{m+r,s}$-graphic by Lemma 3.4(3). Clearly, π is also potentially $A_{m,r,s}$-graphic. □

Lemma 3.8. *Let n be sufficiently large and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n)) + 2$, $n - 2 \geqslant d_1 \geqslant \cdots \geqslant d_{m+r+s-1} \geqslant d_{m+r+s} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n$ and $d_{m+r} \leqslant m + r + s - 2$. Then π is potentially $B_{m,r,s}$-graphic.*

Proof. Take $t = \max\{i \mid d_i \geqslant m+r+s-1\}$. By Lemmas 3.1 and 3.4, $m \leqslant t \leqslant m+r-1$ and $\pi_t = (d_{t+1}-t, \dots, d_{m+r+s}-t, d^{(t)}_{m+r+s+1}, \dots, d^{(t)}_n)$, where $d_{t+1}-t \leqslant m+r+s-2-t$, $2 \leqslant d^{(t)}_{m+r+s+1} \leqslant m+r+s-2$ and $p(\pi_t) = p(\pi) \geqslant \sqrt{\sigma(\pi)}$ is sufficiently large. It is easy to get from Theorem 3.3 that π_t is graphic, and hence π is potentially $A_{t,m+r+s-t}$-graphic by Lemma 3.4(3). ByLemma 3.2, there exists an integer k, $1 \leqslant k \leqslant \min\{r-1, \lceil \frac{s}{2} \rceil - 1\}$ such that $d_{m+r+k} \geqslant m+r+s-1-k$ and $d_{m+r+s} \geqslant m + r + k$. We consider the following two cases.

Case 1: $t - m > (r-1) - k$. In this case, $d_{t+k+1} - t \geqslant d_{m+r+k} - t \geqslant m+r+s-t-(k+1)$ and $d_{m+r+s} - t \geqslant m+r+k-t \geqslant k+1$. Hence, it is easy to see that π_t has a realization H with vertex set $V(H) = \{v_{t+1}, \dots, v_n\}$ such that $d_H(v_i) = d_i - t$ for $t+1 \leqslant i \leqslant m+r+s$ and $d_H(v_i) = d^{(t)}_i$ for $m+r+s+1 \leqslant i \leqslant n$, and H contains $K_{k+1,m+r+s-t-(k+1)}$ as its subgraph, where $\{v_{t+1}, \dots, v_{t+k+1}\}$, $\{v_{t+k+2}, \dots, v_{m+r+s}\}$ is the bipartite partition of the vertex set of $K_{k+1,m+r+s-t-(k+1)}$. By the definition of π_t, π has a realization G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ such that (1) $d_G(v_i) = d_i$ for $1 \leqslant i \leqslant n$, (2) G contains $K_{t,k+1,m+r+s-t-(k+1)}$ as its subgraph, where $\{v_1, \dots, v_t\}$, $\{v_{t+1}, \dots, v_{t+k+1}\}$, $\{v_{t+k+2}, \dots, v_{m+r+s}\}$ is the 3-partite partition of the vertex set of $K_{t,k+1,m+r+s-t-(k+1)}$,(3) the induced subgraph of $\{v_1, \dots, v_t\}$ in G is a clique. Clearly, G also contains $K_{m,r,s}$ as its subgraph, where $\{v_1, \ldots, v_m\}$, $\{v_{m+1}, \ldots, v_{m+r-1-k}, v_{t+1}, \ldots, v_{t+k+1}\}$, $\{v_{m+r-k}, \ldots, v_t, v_{t+k+2}, \ldots, v_{m+r+s}\}$ is the 3-partite partition of the vertex set of $K_{m,r,s}$, and the induced subgraph of $\{v_1, \ldots, v_m\}$ in G is a clique. In other words, π is potentially $B_{m,r,s}$-graphic.

Case 2: $t - m \leqslant (r-1) - k$. In this case, $d_{m+r} - t \geqslant d_{m+r+k} - t \geqslant m + r + s - t - k - 1 \geqslant s$ and $d_{m+r+s} - t \geqslant m + r + k - t \geqslant m + r - t$. Hence, π_t has a realization H with vertex set $V(H) = \{v_{t+1}, \ldots, v_n\}$ such that $d_H(v_i) = d_i - t$ for $t+1 \leqslant i \leqslant m+r+s$ and $d_H(v_i) = d_i^{(t)}$ for $m+r+s+1 \leqslant i \leqslant n$, and H contains $K_{m+r-t,s}$ as its subgraph, where $\{v_{t+1}, \ldots, v_{m+r}\}$, $\{v_{m+r+1}, \ldots, v_{m+r+s}\}$ is the bipartite partition of the vertex set of $K_{m+r-t,s}$. By the definition of π_t, π has a realization G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that (1) $d_G(v_i) = d_i$ for $1 \leqslant i \leqslant n$, (2) G contains $K_{m,r,s}$ as its subgraph, where $\{v_1, \ldots, v_m\}$, $\{v_{m+1}, \ldots, v_{m+r}\}$, $\{v_{m+r+1}, \ldots, v_{m+r+s}\}$ is the 3-partite partition of the vertex set of $K_{m,r,s}$, (3) the induced subgraph of$\{v_1, \ldots, v_m\}$ in G is a clique. In other words, π is potentially $A_{m,r,s}$-graphic. □

We now prove the following upper bound of $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n)$ for sufficiently large n.

Theorem 3.7. *If n is sufficiently large and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geqslant \sigma(\pi(m, r, s, n)) + 2$, then π is potentially $K_{r_1,r_2,\ldots,r_\ell,r,s}$-graphic. In other words, $\sigma(K_{r_1,r_2,\ldots,r_\ell,r,s}, n) \leqslant \sigma(\pi(m, r, s, n)) + 2$ for n sufficiently large.*

Proof. It is enough to prove that π is potentially $B_{m,r,s}$-graphic. Apply induction on m. If $m = 0$, then by Theorems 1.1 and 3.6, π is potentially $B_{0,r,s}$-graphic, i.e., Theorem 3.7 holds for $m = 0$. Assume that Theorem 3.7 holds for $m - 1$ ($\geqslant 0$). We will prove that Theorem 3.7 holds for m. Let $\pi_1' = (d_1', \ldots, d_{n-1}')$ be the residual sequence obtained from π by laying off d_1. Then, π_1' is graphic, and it is easy to check that $\sigma(\pi_1') = \sigma(\pi) - 2d_1 \geqslant \sigma(\pi(m, r, s, n)) + 2 - 2(n-1) = \sigma(\pi(m-1, r, s, n-1)) + 2$. For example, if $3 \leqslant r \leqslant s \leqslant 2r$, s is even and $(r, s, n-m) \in A_1$, then

$$
\begin{aligned}
\sigma(\pi_1') = \sigma(\pi) - 2d_1 &\geqslant \sigma(\pi(m, r, s, n)) + 2 - 2(n-1) \\
&= 2(m-1)n - m(m+1) + f(r, s, n-m) + 4 \\
&= 2(m-1)(n-1) - (m-1)m + f(r, s, n-1-(m-1)) + 2 \\
&= \sigma(\pi(m-1, r, s, n-1)) + 2.
\end{aligned}
$$

Since $n-1$ is also sufficiently large, by the induction hypothesis, π_1' is potentially $B_{m-1,r,s}$-graphic. If $d_1 = n-1$, or there exists an integer k, $m+r+s \leqslant k \leqslant d_1 + 1$ such that $d_k > d_{k+1}$, then $d_2 - 1, \ldots, d_{m+r+s} - 1$ are the $m+r+s-1$ largest terms in π_1'. Thus, π is potentially $B_{m,r,s}$-graphic. So we may assume that

$$
n - 2 \geqslant d_1 \geqslant \cdots \geqslant d_{m+r+s-1} \geqslant d_{m+r+s} = \cdots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \cdots \geqslant d_n.
$$

Lemmas 3.5–3.8 show that π is also potentially $B_{m,r,s}$-graphic. □

Proof of Theorem 1.2. Theorem 1.2 follows immediately from Theorems 2.2 and 3.7. □

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References

[1] C. Berge, Graphs and Hypergraphs, North-Holland Publishing Co, Amsterdam, 1973.

[2] P. Erdős, T. Gallai, Graphs with given degrees of vertices, Math. Lapok 11 (1960) 264–274.

[3] R.J. Gould, M.S. Jacobson, J. Lehel, Potentially *G*-graphical degree sequences, in: Y. Alavi, G. Lick, A.J. Schwenk (Eds.), Combinatorics, Graph Theory and Algorithms, vol. 1, New Issues Press, Kalamazoo MI, 1999, pp. 451–460.

[4] A.E. Kézdy, J. Lehel, Degree sequences of graphs with prescribed clique size, in: Y. Alavi et al. (Eds.), Combinatorics, Graph Theory, and Algorithms, vol. 2, New Issues Press, Kalamazoo MI, 1999, pp. 535–544.

[5] D.J. Kleitman, D.L. Wang, Algorithm for constructing graphs and digraphs with given valences and factors, Discrete Math. 6 (1973) 79–88.
[6] A.R. Rao, An Erdős-Gallai type result on the clique number of a realization of a degree sequence, unpublished.
[7] J.H. Yin, J.S. Li, The smallest degree sum that yields potentially $K_{r,r}$-graphic sequences, Sci. China Ser. A 45 (2002) 694–705.
[8] J.H. Yin, J.S. Li, An extremal problem on potentially $K_{r,s}$-graphic sequences, Discrete Math. 260 (2003) 295–305.
[9] J.H. Yin, J.S. Li, G.L. Chen, A variation of a classical Turán-type extremal problem, European J. Combin. 25 (2004) 989–1002.
[10] J.H. Yin, J.S. Li, G.L. Chen, The smallest degree sum that yields potentially $K_{2,s}$-graphic sequences, Ars. Combin. 74 (2005) 213–222.